FREQUENCY INDEPENDENT ANTENNAS

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Summary

There is a class of antennas whose pattern as well as impedance is practically independent of frequency for all frequencies above a certain value. The general formula for their shape is

$$r = e^{a(\varphi+\varphi_0)}F(\theta)$$

where $r\ \theta\ \varphi$ are the usual spherical coordinates, a and φ_0 are constants and $F(\theta)$ is any function of θ. Assuming a to be positive, φ ranges from $-\infty$ to some finite value which determines the low frequency limit. For such antennas a change of frequency is equivalent to a rotation of the antenna about $\theta=0$. It appears that the pattern converges to the characteristic pattern as the frequency is raised, if a is not ∞, and that the impedance converges to the characteristic impedance for all a.

Antennas Specified by Angles

It is common experience that if all the dimensions of a lossless antenna are increased by a factor K, the pattern and impedance remain fixed if the operating wavelength is also increased by the factor K. In other words, the performance of a lossless antenna is independent of frequency if its dimensions measured in wavelengths are held constant. It follows that if the shape of the antenna were such that it could be specified entirely by angles, its performance would be independent of frequency. The infinite biconical antenna[1] is the most familiar example: it is specified by the angles of the two cones and the angle between their axes. There is, however, an infinite variety of shapes which are completely specified by angles and these form the starting point for the design of frequency independent antennas. They must all extend to infinity (because if they did not they would have at least one characteristic length) and therefore they do not immediately lead to practical designs. The key problem is therefore to determine how rapidly, if at all, the performance of the finite structure converges to that of the infinite structure. Let us, however defer consideration of this question until we have explored some applications of the "angle method" to familiar antennas.

The problem of a typical directional antenna can be illustrated by considering a unipole in front of a plane reflector as shown in Fig. 1: it is specified by the lengths ABCDdℓ One of the major limitations on its pattern and impedance bandwidth is represented by the distance, D, from the unipole to the reflector. By changing the reflector and unipole to coapical cones all dimensions except B and ℓ are replaced by angles. This simple application of the angle method does indeed give significant improvement of the impedance and pattern bandwidth[2]. The impedance is practically constant above a certain frequency but unfortunately the pattern is not. This appears to be typical of most conventional antennas designed according to the angle method. However, there is a whole class of unconventional antennas which have not only an impedance but also a pattern which remains practically constant above a certain frequency. In view of the fact that all experience suggests that the pattern of any antenna develops more and sharper lobes as the frequency is increased, this result is indeed remarkable.

The General Approach

To illustrate the general approach, consider all plane curves which remain essentially the same when scaled to a different unit of length. Such curves can be used to determine the shape of a plane sheet antenna, by taking the input terminals at the common point of intersection of four curves, as illustrated in Fig. 2. It follows then, that the antenna is unchanged when scaled to a different wavelength, provided we add the condition that the terminals stay fixed when the scale is changed. Now the fact that a typical curve remains essentially unchanged by a change of scale implies that the new curve can be made to coincide with the old one by translation and rotation. Since a translation is eliminated by the requirement that the common point remain fixed, the problem is to determine all curves such that a change of scale is equivalent to a rotation. This can be stated symbolically in the form

$$Kr(\varphi) = r(\varphi+C) \tag{1}$$

where $r(\varphi)$ denotes the radius r as a function of the polar angle φ, K is the scale change and C the angle of rotation to which it is equivalent. Thus K depends on C but K and C are independent of φ(and r).

$$\therefore \quad r(\varphi)\frac{dK}{dC} = \frac{\partial r(\varphi+C)}{\partial C} \tag{2}$$

and

$$K\frac{dr(\varphi)}{d\varphi} = \frac{\partial r(\varphi+C)}{\partial\varphi}. \tag{3}$$

But

$$\frac{\partial r(\varphi+C)}{\partial C} = \frac{dr(\varphi+C)}{d(\varphi+C)} = \frac{\partial r(\varphi+C)}{\partial\varphi} \tag{4}$$

$$\therefore \quad r(\varphi)\frac{dK}{dC} = K\frac{dr(\varphi)}{\partial\varphi} \tag{5}$$

or

$$\frac{dr}{d\varphi} = ar \tag{6}$$

where a is independent of φ:

$$a = \frac{1}{K}\frac{dK}{dC}. \tag{7}$$

It follows from (6) that

$$r = r_o e^{a\varphi} \text{ where } r_o \text{ is a constant.} \tag{8}$$

Let $r_o = e^{a\varphi_o}$ where φ_o is a constant (9)

or $a\varphi_o = \ln r_o$. (10)

Then $r = e^{a(\varphi+\varphi_o)}$ (11)

or $\varphi+\varphi_o = \ln r^{1/a}$. (12)

We recognize (11) or (12) as the formula for an equiangular spiral: it contains the two parameters, a, which represents the rate of expansion, and φ_o, which represents the orientation. Thus the shape of all plane sheet frequency independent antennas must be defined by equiangular spirals. (Note that when $1/a = 0$ the spiral degenerates into the straight line $\varphi = -\varphi_o$.) Theoretically it might appear that we could obtain four curves such as shown in Fig. 2 by selecting four different combinations of a and φ_o, but it is easily verified that unless a is the same for all, such curves overlap at infinitesimal values of r, thus placing a short circuit across the terminals. It is therefore necessary to choose four different φ_o with the same a.

The general problem is to find all *surfaces* which have the property that a change in the unit of length is equivalent to a certain rotation. Then if we construct a metal antenna whose surface is one of these surfaces, its performance will be the same at all wavelengths except for a rotation of the coordinate system. This problem is much more difficult to analyze than the plane case we have just considered, and we shall simply give an outline of the method. The typical surface can be represented by the formula

$$r = f(\theta\varphi) \tag{13}$$

where r θ and φ are the usual spherical coordinates. After a rotation the surface is represented by

$$r = f(\theta'\varphi') \tag{14}$$

where $\theta'\varphi'$ are related to $\theta\varphi$ by the rotation. Then the condition that the rotation be equivalent to a uniform expansion is expressed by the equation

$$Kf(\theta\varphi) = f(\theta'\varphi') \text{ for all } \theta\varphi \tag{15}$$

where K is independent of $\theta\varphi$ but depends on the parameters of the rotation.

There are two difficulties in trying to find the function f from (15): one is that the relation between $\theta'\varphi'$ and θ φ is very involved and the other is that the rotation depends on three parameters, such as the two spherical coordinates which specify the axis of rotation, and the angle of rotation about the axis. To express the transformation from θ φ to $\theta'\varphi'$ explicitly let $\underline{u}$ represent a unit vector in the $(\theta\varphi)$ direction, i.e.,

$$u_x = \sin\theta\cos\varphi \quad u_y = \sin\theta\sin\varphi \quad u_z = \cos\theta \tag{16}$$

Similarly let $\underline{u}'$ be defined for $\theta'\varphi'$.

Then the transformation is expressed by

$$\underline{u}' = T\,\underline{u} \tag{17}$$

where T is a matrix which is independent of θ, φ, θ', and φ': T is a function of the three parameters, a_1, a_2, a_3, of the rotation. We can now obtain a set of partial differential equations for the function f by differentiating (15) in a way analagous to the determination of the function r from (1). Let r and r' denote $f(\theta\varphi)$ and $f(\theta'\varphi')$.

Differentiation of (15) with respect to θ and φ gives

$$K\,\underline{\alpha} = M\,\underline{\alpha}' \text{ (2 equations)} \tag{18}$$

where

$$\alpha_i = \frac{\partial r}{\partial\theta_i} \quad i = 1,2 \ (\theta_1 = \theta \quad \theta_2 = \varphi) \tag{19}$$

and

$$M_{ij} = \frac{\partial\theta'_i}{\partial\theta_i} \tag{20}$$

Differentiation of (15) with respect to a_i gives

$$r\,\underline{\beta} = S\,\underline{\alpha}' \text{ (3 equations)} \tag{21}$$

where

$$\beta_i = \frac{\partial K}{\partial a_i} \qquad i = 1, 2, 3 \tag{22}$$

$$S_{ij} = \frac{\partial\theta'_j}{\partial a_i} \qquad i = 1, 2, 3 \quad j = 1, 2. \tag{23}$$

Differentiation of (17) with respect to θ_i gives

$$A\tilde{T} = MA' \text{ (6 equations)} \tag{24}$$

where

$$A_{ij} = \frac{\partial u_j}{\partial\theta_i},\ A'_{ij} = \frac{\partial u'_j}{\partial\theta'_i} \quad j=1,2,3 \ i=1,2 \tag{25}$$

($\tilde{T}$ is T transposed, i.e., $\tilde{T}_{ij}=T_{ji}$, $T\tilde{T}=1$).

Differentiation of (17) with respect to a_i gives

$$N=SA' \text{ (9 equations)} \tag{26}$$

where

$$N_{ij} = \sum_{K=1,2,3} \frac{\partial T_{jk}}{\partial a_i} u_k \quad i = 1,2,3 \quad j = 1,2,3. \quad (27)$$

The next step is to eliminate $\underline{\alpha}'$ from (18) and (21) using (24) and (26) to evaluate M and S. The algebra can be greatly simplified if we choose the coordinates so that the z axis is the axis of rotation *corresponding to a particular expansion K*: this does not imply that the axis of rotation is fixed for all K. Thus in (27) we perform the differentiation before substitution of the particular values of $a_1a_2a_3$ which correspond to this choice of axes. With this simplification, which does not involve any loss of generality, it is practical to derive a set of equations of the form

$$r\ \underline{\beta} = P\ \underline{\alpha} \text{ (3 equations)} \quad (28)$$

where P is a known (3×2) matrix expressed in terms of $\theta\varphi$ and the angle of rotation. Recalling the definitions of $\underline{\alpha}$ and $\underline{\beta}$ we see that (28) constitutes a set of partial differential equations for K as a function of $a_1a_2a_3$ and r as a function of $\theta\varphi$. It turns out that the only non-trivial solution is the simple case where the axis of rotation is fixed, i.e. K depends on only one rotation parameter. The problem now is essentially the same as the plane case. The solution can be written in the form

$$r = e^{a(\varphi+\varphi_o)}\ F(\theta) \quad (29)$$

where in principle $F(\theta)$ can be any function of θ. The shapes represented by (29) can be very complicated because in general an increase of 2π in φ does not give the same r: as φ ranges from $-\infty$ to ∞ the surface weaves around through all space. Fig. 3 illustrates a simple example which gives a practical antenna design. Fig. 4 illustrates the case where $\ln F(\theta)$ is periodic in θ with period $2\pi a$. This gives a simple surface like a screw thread which is uniformly expanded in proportion to the distance from the origin: an increase of 2π in φ is equivalent to moving one turn along the screw. The plane antennas considered in the accompanying paper by R.H. DuHamel and D.E. Isbell[3] represent a cross section through the axis of this kind of antenna.

The Pattern

To examine the question of pattern convergence we make use of a characteristic of the field which depends on the fact that a rotation is equivalent to an expansion. Since we are dealing with the frequency independent mode, the analysis of the field can be simplified by considering the static or DC case. For this case it has been shown by P.E. Mast that for the plane antenna the field is a function of the two variables $S = re^{-a\varphi}$ and θ, rather than the three variables $r\theta\varphi$. (Note that S is constant on any equiangular spiral in the family characterized by the parameter a). This result implies that the direction of the current on the antenna is along the lines S = constant, i.e., the currents follow a spiral flow. The simplest way to analyze the pattern is to make use of the intuitive idea of coupling between adjacent turns of this spiral flow. Assuming to begin with that the AC current distribution is like the DC distribution traveling with the free space phase velocity, we see that a "resonance" will build up in the current distribution where the mean circumference of the spiral is about one wavelength. This resonance will be highly damped by radiation so that most of the current is dissipated in the first resonance. The pattern due to the first resonance is therefore almost the same as the pattern due to the infinite structure. Admittedly this argument is weak but it is not altogether worthless because it does give a simple way of estimating the pattern, which agrees satisfactorily with measurements on finite antennas. It shows that the condition for pattern convergence is that the current flow be spiral rather than rectilinear, i.e., that the parameter a should not be infinite.

These theoretical conjectures are borne out in practice. The case $a = \infty$ represents the biconical type of antenna for which the pattern does not converge. Various cases for $a \neq \infty$ have been investigated by J.D. Dyson[4] and R.L. Carrel[5] and convincing evidence of the constancy of the pattern over a 20:1 frequency range has been obtained. Note that a change of frequency is equivalent to a rotation. The pattern at one frequency, f_1, is the same as the pattern at another frequency, f_2, if the coordinate system is rotated about the $\theta = 0$ axis through an angle $1/a \ln f_1/f_2$: the pattern scans around the $\theta = 0$ axis at a rate which depends on a. If measurements are made in a fixed plane, the pattern in general varies with frequency and the frequency independent nature of the complete spherical pattern may be missed in such measurements.

The Impedance

The convergence of the input impedance to a constant value as the frequency is increased is familiar in connection with the biconical antenna. It has been confirmed in all cases of the general type represented by (29) which have been investigated.[4,5]. The determination of the characteristic impedance is an important problem in connection with frequency independent antennas. Schelkunoff's well known formula for the characteristic impedance of two coaxial cones[1] is

$$377 \ln \left(\tan \frac{A}{2} \operatorname{Cot} \frac{B}{2}\right)$$

where A and B are the cone angles measured from the common axis. The formula for two inclined cones is[6]

$$377 \left\{ \tanh^{-1} \left[\frac{\tan \frac{\varphi_1}{2}}{\tan \frac{\psi}{2}} \right] + \tanh^{-1} \left[\frac{\tan \frac{\varphi_2}{2}}{\tan \frac{\psi}{2}} \right] \right\}$$

where φ_1 φ_2 and ψ are defined in Fig. 5.

The formula for a symmetrical plane sheet antenna consisting of two triangles with a common apex at the terminals[7] is

$$189 \frac{K(\cos\psi)}{K(\sin\psi)}$$

where K represents the elliptic integral defined by $K(x) = \int_0^1 \frac{dt}{(1-t^2)(1-x^2t^2)}$ and ψ represents the half angle of the triangular strips measured from the common axis.

In connection with the impedance we should note an interesting property which was pointed out by Mushiake in one of the Tohoku University reports. It is that the impedance of any plane sheet antenna whose shape is the same as the shape of its complement (except for a trivial change of coordinates) is independent of frequency and equal to 60π = 189 ohms. The complementary antenna is defined as the portion of a metal plane which is not covered by the original antenna: when the antenna and its complement are fitted together they completely cover the whole plane without overlapping. The constant impedance of a "self-complementary" antenna follows from the relation[8]

$$Z_1 Z_2 = (60\pi)^2$$

between the impedance Z_1 of the antenna and the impedance Z_2 of its complement. Fig. 6 gives some examples of self-complementary shapes.

Pseudo Frequency Independent Antennas

The idea of a pseudo frequency independent antenna is illustrated by the horn antenna shown in Fig. 7. It consists of metal sheet perforated by holes of uniformly expanding size: any hole is exactly like its neighbor on the left except for a fixed expansion. The idea is that the effective size of the horn remains roughly independent of wavelength because the metal sheet becomes approximately transparent once the holes become greater than about half a wavelength square. More precisely, it can be seen that if the horn started from a point and extended to infinity, it would "look" the same to any two wavelengths whose ratio was equal to the expansion factor. Some interesting examples of pseudo frequency independent antennas are described in the accompanying paper by R.H. DuHamel and D.E. Isbell.

Acknowledgement

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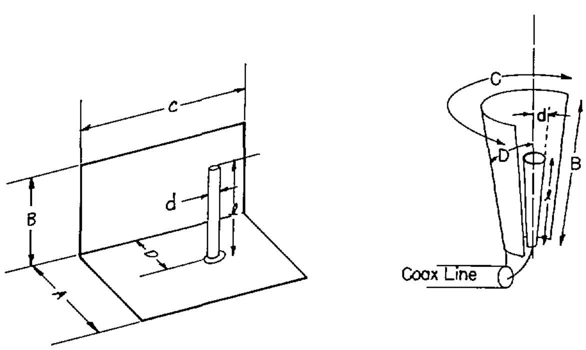

Fig. 1

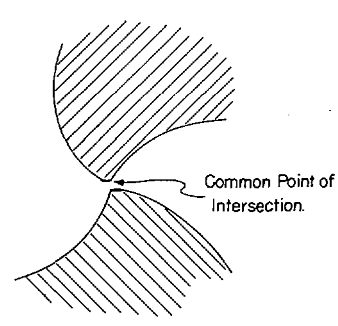

Fig. 2

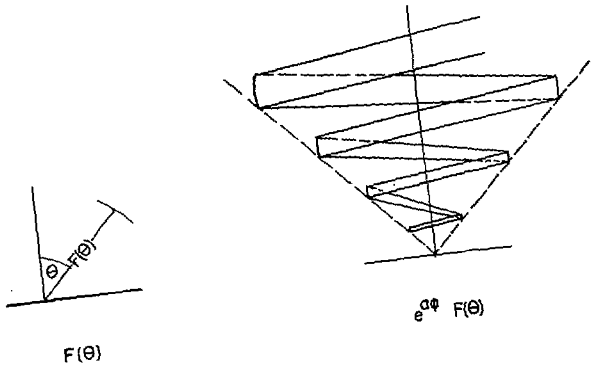

Fig. 3

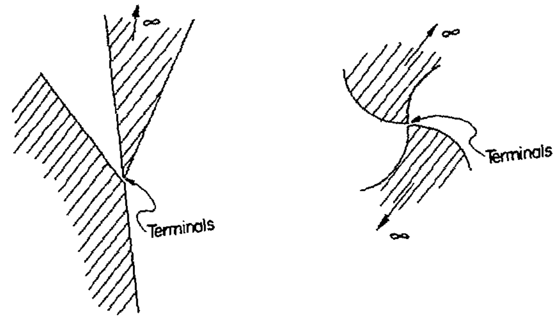

Fig. 6

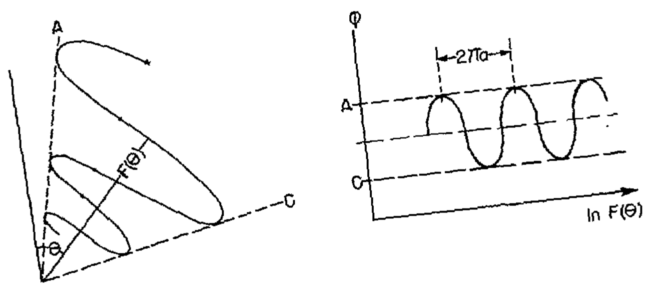

Fig. 4

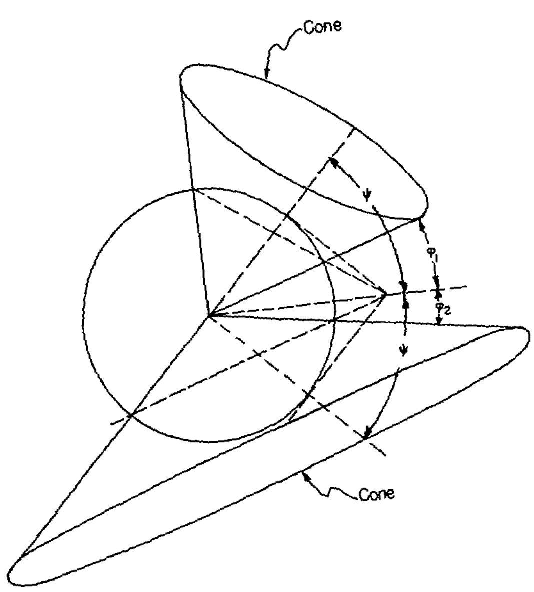

Fig. 5

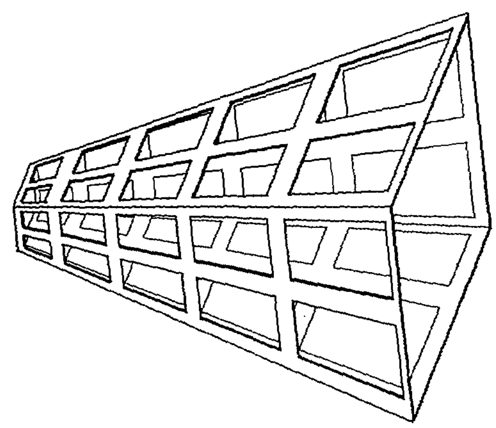

Fig. 7