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Thickness Vibrations of Piezoelectric Oscillating Crystals

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For several years, it has been well known that, in the natural vibration of a piezoelectric oscillating crystal, there is a certain period proportional only to the thickness, but, so far as we are aware, there has been no concrete explanation for this vibration nor how its period is connected with the physical constants of the medium. The present paper shows that such a vibration, named *thickness vibration* for brevity, is due to the standing wave produced by interference of plane waves incident to and reflected from the plane boundary surfaces of the medium, and verifies the theoretical results by several examples. There are three normal modes in the thickness vibration, and the corresponding frequencies are given in first approximation by a representative formula: $P/2\pi=(q/2a)\ (c/\rho)^{1/2}$, where q is any integer, a the thickness of the plate, ρ the density and c a certain adiabatic elastic constant depending upon the orientation of the plate with respect to the crystallographic axes of the medium. It is not always possible to excite all normal modes of vibration piezoelectrically. The electrically measured natural frequency of the thickness vibration is always a little higher than that calculated by the formula. Adiabatic elastic constants of piezoelectric crystals may be determined in first approximation by measurement of the frequencies of thickness vibrations of plates prepared from given crystals.

I. Introduction

As IS well known, a thin piezoelectric crystal plate has in general many modes of vibration, but the shortest period of fundamental vibration is always dependent only upon the thickness of the plate. We shall name this vibration the *thickenss vibration.*

Although many papers have been published upon the vibration of piezoelectric crystal, none has yet pointed out clearly, as far as we are aware, that the thickness vibration of thin crystal plate is a mere result of standing waves produced by the interference of plane waves, the wave surfaces of which are parallel to the boundary surfaces.

II. Equations of Motions in Crystalline Media

The fundamental period of the thickness vibration of a very thin crystal plate is dependent almost entirely upon the thickness of the plate, so that we may consider the extent of the surfaces to be infinite.

The free vibration of a thin crystal plate having two parallel plane boundary surfaces is due to the interference of plane waves incident to and reflected from those boundary surfaces. In other words, in the thickness vibration of the crystal plate bounded by the two infinite parallel planes $x=0$ and $x=a$, the displacement at any point in the medium is dependent only upon x and t, and not upon y and z. Therefore the general equations of motions in a crystalline medium,

$$\left.\begin{aligned} \partial X_x/\partial x + \partial X_y/\partial y + \partial X_z/\partial z &= \rho\partial^2 u/\partial t^2, \\ \partial Y_x/\partial x + \partial Y_y/\partial y + \partial Y_z/\partial z &= \rho\partial^2 v/\partial t^2 \\ \partial Z_x/\partial x + \partial Z_y/\partial y + \partial Z_z/\partial z &= \rho\partial^2 w/\partial t^2, \end{aligned}\right\} \tag{1}$$

where

$$\left.\begin{aligned} X_x &= c_{11}e_{xx} + c_{12}e_{yy} + c_{13}e_{zz} + c_{14}e_{yz} + c_{15}e_{zx} + c_{16}e_{xy}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ X_y &= c_{61}e_{xx} + c_{62}e_{yy} + c_{63}e_{zz} + c_{64}e_{yz} + c_{65}e_{zx} + c_{66}e_{xy}, \\ c_{hk} &= c_{kh}, \\ e_{xx} &= \frac{\partial u}{\partial x}, \quad e_{yy} = \frac{\partial v}{\partial y}, \quad e_{zz} = \frac{\partial w}{\partial z}, \\ e_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \; e_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \quad e_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \end{aligned}\right\} \tag{2}$$

u, v, w, displacement in the direction of x, y, z respectively, ρ, density of the medium, reduce to the very simple forms:

$$\frac{\partial X_x}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2}, \quad \frac{\partial Y_x}{\partial x} = \rho\frac{\partial^2 v}{\partial t^2}, \quad \frac{\partial Z_x}{\partial x} = \rho\frac{\partial^2 w}{\partial t^2}, \tag{3}$$

or further

$$\left.\begin{aligned} (\partial^2/\partial x^2)(c_{11}u + c_{16}v + c_{15}w) &= \rho\partial^2 u/\partial t^2, \\ (\partial^2/\partial x^2)(c_{16}u + c_{66}v + c_{56}w) &= \rho\partial^2 v/\partial t^2, \\ (\partial^2/\partial x^2)(c_{15}u + c_{56}v + c_{55}w) &= \rho\partial^2 w/\partial t^2. \end{aligned}\right\} \tag{4}$$

Now let l, m, n and c be so chosen that

$$\left.\begin{aligned} lc_{11} + mc_{16} + nc_{15} &= lc \\ lc_{16} + mc_{66} + nc_{56} &= mc \\ lc_{15} + mc_{56} + nc_{55} &= nc \end{aligned}\right\} \tag{5}$$

and put

$$\xi = lu + mv + nw, \tag{6}$$

then, multiplying the three equations of (4) by l, m, n and adding together, we have in the most compact form:

$$\partial^2\xi/\partial t^2 = (c/\rho)(\partial^2\xi/\partial x^2). \tag{7}$$

Eliminating l, m, n between the three equations of (5), we have

$$\begin{vmatrix} c_{11} - c & c_{16} & c_{15} \\ c_{16} & c_{66} - c & c_{56} \\ c_{15} & c_{56} & c_{55} - c \end{vmatrix} = 0 \tag{8}$$

a cubic equation determining c; let its roots, which are proved to be necessarily real and positive,[1] be c_1, c_2, c_3. When c_1 is substituted in any two of the foregoing equations, the ratios of $l:m:n$ can be derived; let them be denoted

[1] Lord Kelvin, Baltimore Lectures, London, 1904.

by $l_1:m_1:n_1$, and suppose the corresponding value of ξ to be ξ_1 with similar expressions for the other values of c. Then ξ_1, ξ_2, and ξ_3 are always proved to be perpendicular to each others.

The solution of (7) is a very simple matter. To determine the normal mode of vibration we must assume that ξ varies as a harmonic function of the time t

$$\xi \propto e^{ipt}. \tag{9}$$

Then as a function of x, ξ must satisfy

$$d^2\xi/dx^2 + (p^2\rho/c)\xi = 0 \tag{10}$$

of which the complete integral is

$$\xi = A \cos\left[px(\rho/c)^{1/2}\right] + B \sin\left[px(\rho/c)^{1/2}\right] \tag{11}$$

where A and B are independent of x.

Now when both boundary surfaces are free from tractions,

$$X_x = Y_x = Z_x = 0 \text{ or } \partial\xi/\partial x = 0 \text{ at } x = 0 \text{ and } x = a, \tag{12}$$

we get

$$B = 0 \text{ and } \sin\left[pa(\rho/c)^{1/2}\right] = 0 \tag{13}$$

from which

$$p/2\pi = (q/2a)(c/\rho)^{1/2} \tag{14}$$

q being integral.

Accordingly, the normal mode is given by equation of the form

$$\xi = A \cos(q\pi x/a)\cdot e^{ipt} \tag{15}$$

where A is an arbitrary constant, which may be determined in the usual manner, when the initial value of ξ is given.

Similarly if the medium be bounded by two parallel planes $y=0$ and $y=a$, we may replace (5), (8) and (15) by

$$lc_{66} + mc_{26} + nc_{46} = lc;\ lc_{26} + mc_{22} + nc_{24} = mc;\ lc_{46} + mc_{24} + nc_{44} = nc. \tag{16}$$

$$\begin{vmatrix} c_{66} - c & c_{26} & c_{46} \\ c_{26} & c_{22} - c & c_{24} \\ c_{46} & c_{24} & c_{44} - c \end{vmatrix} = 0 \tag{17}$$

$$\xi = A \cos(q\pi y/a)\cdot e^{ipt} \tag{18}$$

and if the medium be bounded by $z=0$ and $z=a$, by

$$lc_{55} + mc_{45} + nc_{35} = lc;\ lc_{45} + mc_{44} + nc_{34} = mc;\ lc_{35} + mc_{34} + nc_{33} = nc. \tag{19}$$

$$\begin{vmatrix} c_{55} - c & c_{45} & c_{35} \\ c_{45} & c_{44} - c & c_{34} \\ c_{35} & c_{34} & c_{33} - c \end{vmatrix} = 0 \tag{20}$$

$$\xi = A \cos(q\pi z/a)\cdot e^{ipt}. \tag{21}$$

It is well known that we can excite one or more of the free thickness vibrations by applying an alternating electric field of suitable frequency in the direction normal to the boundary surfaces of crystal, provided the crystal plate is prepared so that it is piezoelectrically active; and also that it is sometimes possible to sustain a continuous oscillation by properly employing the thermionic tube circuit. The ease of maintenance depends considerable upon the medium and the orientation of the boundary surfaces.

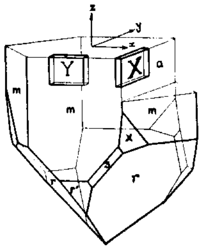

Fig. 1.

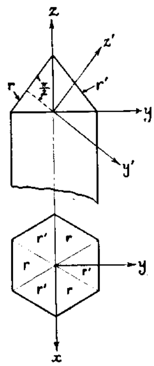

Fig. 2.

III. Thickness Vibration of Quartz Plate

Of the piezoelectric crystals, quartz is the most familiar to high-frequency engineers, so that we may first verify the theory with this medium. In quartz, if the coordinate axes be chosen as in Fig. 1, the adiabatic elastic constants[2] and density are as follows:

$$\left.\begin{aligned} c_{11} &= 85.45 \times 10^{10}\,\text{dynes/cm}^2, & c_{12} &= 7.26 \times 10^{10}\,\text{dynes/cm}^2,\\ c_{33} &= 105.67 \times 10^{10}\,\text{dynes/cm}^2, & c_{13} &= 14.37 \times 10^{10}\,\text{dynes/cm}^2,\\ c_{44} &= 57.09 \times 10^{10}\,\text{dynes/cm}^2, & -c_{14} &= 16.87 \times 10^{10}\,\text{dynes/cm}^2,\\ & & \rho = 2.654\,\text{grams/cm}^3, & \end{aligned}\right\} \quad (22)$$

while the relations between applied electric fields and strains are:

$$\left.\begin{aligned} e_{xx} &= d_{11}E_x, & e_{zx} &= -d_{14}E_y,\\ e_{yy} &= -d_{11}E_x, & e_{xy} &= -2d_{11}E_y,\\ e_{yz} &= d_{14}E_x, & &\\ d_{11} &= 6.45 \times 10^{-8}\ \text{C.G.S. units}, & &\\ -d_{14} &= 1.45 \times 10^{-8}\ \text{C.G.S. units}. & & \end{aligned}\right\} \quad (23)$$

(1) In X-cut quartz plate designated by X in Fig. 1, if the boundary surfaces be expressed by $x=0$ and $x=a$, (8) reduces to

$$\begin{vmatrix} c_{11}-c & 0 & 0\\ 0 & \tfrac{1}{2}(c_{11}-c_{12})-c & c_{14}\\ 0 & c_{14} & c_{44}-c \end{vmatrix} = 0 \quad (24)$$

[2] W. Voigt, Lehrbuch der Kristallphysik, 754 and 789 (1928).

or

$$\left.\begin{aligned} c_1 &= c_{11} = 85.45 \times 10^{10}\ \text{dynes/cm}^2, \\ c_2, c_3 &= \tfrac{1}{4}(c_{11} - c_{12}) + \tfrac{1}{2}c_{44} \pm \left[\left\{\tfrac{1}{4}(c_{11} - c_{12}) - \tfrac{1}{2}c_{44}\right\}^2 + c_{14}^2\right]^{1/2} \\ c_2 &= 67.22 \times 10^{10}\ \text{dynes/cm}^2, \\ c_3 &= 28.98 \times 10^{10}\ \text{dynes/cm}^2, \end{aligned}\right\} \quad (25)$$

and the ratio $l{:}m{:}n$ corresponding to these values:

$$\left.\begin{aligned} l_1{:}m_1{:}n_1 &= 1{:}0{:}0, \\ l_2{:}m_2{:}n_2 &= 0{:}c_2 - c_{44}{:}c_{14} = 0{:}-0.60{:}1, \\ l_3{:}m_3{:}n_3 &= 0{:}c_3 - c_{44}{:}c_{14} = 0{:}1.67{:}1. \end{aligned}\right\} \quad (26)$$

The normal modes of vibration are given by the following equations:

$$\xi = A \cos (q\pi x/a) \cdot e^{ipt}, \quad (27)$$

where $\xi_1 = u$; $\xi_2 = -0.60\ v + w$; and $\xi_3 = 1.67\ v + w$; while the electric field in the direction x cannot excite the displacement ξ_2 and ξ_3 as is seen from (23), so that the period of vibration is

$$p/2\pi = (q/2a)(c_{11}/\rho)^{1/2} = q/a \times 0.2837 \times 10^6\ \text{cycles/sec.}, \quad (28)$$

and the mode of vibration is

$$u = A \cos (q\pi x/a) e^{ipt} \quad (29)$$

that is, the pure longitudinal vibration along the normal to the surfaces.

As an experimental example, a quartz oscillating crystal $0.0922 \times 2.610 \times 2.684\ \text{cm}^3$ (the normal to the major surfaces inclines 7′ the axis x) had the fundamental frequency of thickness vibration 3.10×10^6 cycles/sec., from from which we get

$$p/2\pi = (1/a) + 0.0922 \times 3.10 \times 10^6 = (1/a) \times 0.286 \times 10^6\ \text{cycles/sec.}, \quad (30)$$

which is seen to be very near upon the theoretical result (28).

(2) In Y-cut quartz plate designated by Y in Fig. 1, if the boundary surfaces be expressed by $y = 0$ and $y = a$, (17) reduces to

$$\begin{vmatrix} \tfrac{1}{2}(c_{11} - c_{12}) - c & 0 & 0 \\ 0 & c_{11} - c & -c_{14} \\ 0 & -c_{14} & c_{44} - c \end{vmatrix} = 0 \quad (31)$$

or

$$\left.\begin{aligned} c_1 &= \tfrac{1}{2}(c_{11} - c_{12}) = 39.10 \times 10^{10}\ \text{dynes/cm}^2, \\ c_2, c_3 &= \tfrac{1}{2}(c_{11} + c_{44}) \pm \left[\tfrac{1}{4}(c_{11} - c_{44})^2 + c_{14}^2\right]^{1/2} \\ c_2 &= 93.31 \times 10^{10}\ \text{dynes/cm}^2, \\ c_3 &= 49.23 \times 10^{10}\ \text{dynes/cm}^2, \end{aligned}\right\} \quad (32)$$

and the ratio $l{:}m{:}n$ corresponding to these values:

$$
\left.\begin{aligned}
l_1:m_1:n_1 &= 1:0:0 \\
l_2:m_2:n_2 &= 0:c_{44}-c_2:c_{14} = 0:2.15:1 \\
l_3:m_3:n_3 &= 0:c_{44}-c_3:c_{14} = 0:-0.47:1
\end{aligned}\right\} \tag{33}
$$

The normal modes of vibration are given by

$$\xi = A\cos(q\pi y/a)\cdot e^{ipt} \tag{34}$$

where $\xi_1=u$, $\xi_2=2.15\,v+w$, and $\xi_3=-0.47\,v+w$, of which ξ_2 and ξ_3 cannot be excited piezoelectrically by the electric field in the direction of y axis as is clear from (23), so that the displacement is given by

$$u = A\cos(q\pi y/a)\cdot e^{ipt} \tag{35}$$

which shows that the vibration is of a pure shear as is pointed out by Professor Cady.[3]

The frequency of vibration is

$$\frac{p}{2\pi} = \frac{q}{2a}\left(\frac{c_{11}-c_{12}}{2\rho}\right)^{1/2} = \frac{q}{a}\times 0.1919\times 10^6 \text{ cycles/sec.}, \tag{36}$$

while in the experiment with a sample $0.1909\times3.200\times3.304$ cm^3 (the normal to the major surfaces inclines $6'$ to the axis y) the fundamental frequency of thickness vibration is 1.02×10^6 cycles/sec., that is

$$p/2\pi = 1/a + 0.1909\times1.02\times10^6 = 1/a\times0.195\times10^6 \text{ cycles/sec.}, \tag{37}$$

which is also seen to be very near upon the theoretical result (36).

(3) In quartz plate cut parallel to a surface of positive rhombohedron designated by r in Fig. 1 (say R-cut plate[4]), the vibration is observed to be extremely vigorous and the third harmonic vibration ($q=3$) can often be sustained by Pierce circuit,[5] in which an oscillating crystal plate is placed between grid and filament of a three-electrode thermionic tube.

To obtain periods and normal modes of thickness vibration, we may first transfer the coordinate axes. Let x', y', z', be so chosen that the direction cosines of them referred to the original axes be expressed by the orthogonal scheme

	x	y	z
x'	1	0	0
y'	0	$\cos\theta$	$-\sin\theta$
z'	0	$\sin\theta$	$\cos\theta$

(38)

where

$$\theta = \arctan(3^{1/2}/2.200), \tag{39}$$

[3] W. G. Cady, A shear mode of crystal vibration, Phys. Rev. **29,** 617A (1927).

[4] I. Koga, R-cut quartz oscillating crystal plate and the harmonic vibration, Supplementary Issue, Jour. I.E.E. (Japan) to be issued in April (1932).

[5] G. W. Pierce, Piezoelectric crystal resonators and crystal oscillators applied to the precision calibration of wave meters, Proc. A.A.A.S. **59,** 81–106 (1923).

then the axis y' becomes normal to the surface r as shown in Fig. 2. If the elastic constants referred to the new axes be denoted by c'_{hk}, values of c's in (14) are the roots of

$$\begin{vmatrix} c_{66}' - c & 0 & 0 \\ 0 & c_{22}' - c & c_{24}' \\ 0 & c_{24}' & c_{44}' - c \end{vmatrix} = 0 \qquad (40)$$

or

$$\left.\begin{aligned} c_1 = c_{66}' &= c_{44}\sin^2\theta + \tfrac{1}{2}(c_{11} - c_{12})\cos^2\theta - c_{14}\sin 2\theta \\ &= 62.39 \times 10^{10}\ \text{dynes/cm}^2, \\ c_2,\ c_3 &= \tfrac{1}{2}(c_{22}' + c_{44}') \pm \left[\tfrac{1}{4}(c_{22}' - c_{44}')^2 + c_{24}'^2\right]^{1/2} \end{aligned}\right\} \qquad (41)$$

and the ratio $l{:}m{:}n$ corresponding to these values:

$$\left.\begin{aligned} l_1{:}m_1{:}n_1 &= 1{:}0{:}0 \\ l_2{:}m_2{:}n_2 &= 0{:}c_{44}' - c_2{:}c_{24}' \\ l_3{:}m_3{:}n_3 &= 0{:}c_{44}' - c_3{:}c_{24}' \end{aligned}\right\} \qquad (42)$$

The normal modes of vibration are given by

$$\xi = A\cos(q\pi y'/a)\cdot e^{ipt} \qquad (43)$$

where $\xi_1 = u'$, $\xi_2 = (c_{44}' - c_2)\,v' + c_{24}'w'$, $\xi_3 = (c_{44}' - c_3)\,v' + c_{24}'w'$. On the other hand, the relations (24) transform into the following relations:

$$\left.\begin{aligned} e_{x'x'} &= d_{11}E_{x'}, \\ e_{y'y'} &= -\cos\theta(d_{11}\cos\theta + d_{14}\sin\theta)E_{x'}, \\ e_{z'z'} &= -\sin\theta(d_{11}\sin\theta + d_{14}\cos\theta)E_{x'}, \\ e_{y'z'} &= (d_{11}\sin 2\theta + d_{14})E_{x'}, \\ e_{z'x'} &= -(d_{14}\cos\theta - 2d_{11}\sin\theta)(E_{y'}\cos\theta + E_{z'}\sin\theta), \\ e_{x'y'} &= (d_{14}\sin\theta - 2d_{11}\cos\theta)(E_{y'}\cos\theta + E_{z'}\sin\theta), \end{aligned}\right\} \qquad (44)$$

from which we see that we cannot excite the displacements ξ_2 and ξ_3 by the electric field along the axis y', consequently the displacement is given by

$$u' = A\cos(q\pi y'/a)\cdot e^{ipt} \qquad (45)$$

which shows that the mode of vibration is of a pure shear. The frequency of vibration is

$$p/2\pi = (q/2a)(c_{66}'/\rho)^{1/2} = q/a \times 0.2424 \times 10^6\ \text{cycles/sec.}, \qquad (46)$$

while in the experiment with a sample $0.1002 \times 2.592 \times 2.762\ \text{cm}^3$ (the normal to the major surfaces inclines within $1'$ to the axis y') the fundamental frequency of thickness vibration is 2.47×10^6 cycles/sec., that is

$$p/2\pi = 1/a \times 0.1002 \times 2.47 \times 10^6 = 1/a \times 0.247 \times 10^6\ \text{cycles/sec.}, \qquad (47)$$

which is also seen to be very near the theoretical result (46).

There have been published several papers[6] on the excitation of harmonic vibrations in a resonator, but as far as we are aware, very little work has been done on the harmonics of the thickness vibration of a crystal. The present crystal plate can often be sustained in its first and third harmonic vibrations separately by the Pierce circuit above mentioned. Fig. 3 shows the characteristic features of the oscillator. The higher frequency of the vibrations is easily observed by a heterodyne frequency-meter to be just three times that of the lower.

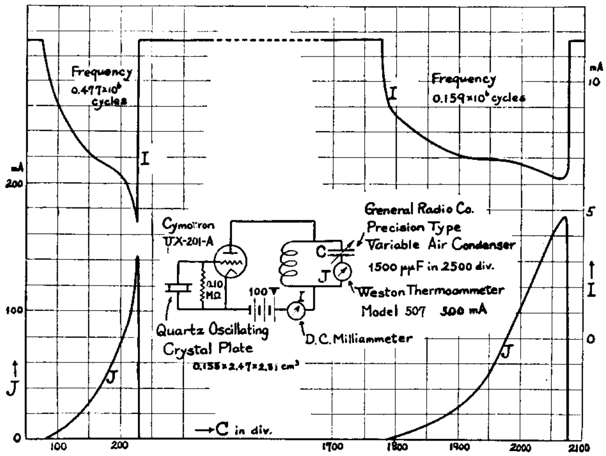

Fig. 3.

Examples of dimensions which are suited for the third harmonic vibrations are:

Dimensions (cm^3)	Approx. frequency of vibrations (cycles/sec.)
$0.093 \times 2.56 \times 2.57$	2.66×10^6 and $2.66 \times 10^6 \times 3$
$0.114 \times 2.09 \times 2.70$	2.17×10^6 and $2.17 \times 10^6 \times 3$
$0.155 \times 2.47 \times 2.81$	1.59×10^6 and $1.59 \times 10^6 \times 3$
$0.185 \times 2.77 \times 2.98$	1.34×10^6 and $1.34 \times 10^6 \times 3$.

(48)

(Inclinations of the major surfaces to the $x'z'$ plane are generally not greater than $10'$).

(4) If the boundary surfaces are parallel to surfaces of a negative rhombohedron designated by r' in Fig. 1, normal modes and frequencies of vibration can be studied as in the previous case. The only difference is the reversal of the sign of θ in Fig. 2. Thus

$$\begin{aligned} c_{66}' &= c_{44}\sin^2\theta + \tfrac{1}{2}(c_{11} - c_{12})\cos^2\theta + c_{14}\sin 2\theta \\ &= 29.59 \times 10^{10} \text{ dynes/cm}^2, \end{aligned} \tag{49}$$

and

$$p/2\pi = (q/2a)(c_{66}'/\rho)^{1/2} = q/a \times 0.1670 \times 10^6 \text{ cycles/sec.} \tag{50}$$

[6] E. Giebe and A. Scheibe, Sichtbarmachung von hochfrequenten Longitudinalschwingungen piezo-elektrischer Kristallstaebe, Zeits. f. Physik **33**, 335 (1925).

The displacement

$$u' = A \cos (q\pi y'/a) \cdot e^{ipt} \tag{51}$$

gives a pure shear vibration.

From the experiment with a sample $0.0765 \times 2.76 \times 3.19$ cm^3 (the normal to the major surfaces inclines $5'$ to the axis y') the frequency of vibration is 2.20×10^6 cycles/sec., that is

$$p/2\pi = 1/a \times 0.0765 \times 2.20 \times 10^6 = 1/a \times 0.168 \times 10^6 \text{ cycles/sec.,} \tag{52}$$

which is very near the theoretical result (50).

It is seen above that the frequency of vibration of a piezoelectric oscillating crystal is always a little higher than that calculated from the elastic constants. This small difference must be, to a certain extent, due to the finiteness of the boundary surfaces of plate, but it is not overlooked that the frequency of pure mechanical vibration is different from that as a piezoelectric oscillating crystal. In fact, when the crystal plate is excited to vibrate with its natural vibration, the equivalent elastic constants become slightly greater than that which is determined elastostatically. For example, by a strain e_{xx}, besides the stress $c_{11}e_{xx}$, the polarization of $\epsilon_{11}e_{xx}$ is produced, ϵ_{11} being the piezoelectric constant, so that the crystal, being acted upon by the electric field $4\pi\epsilon_{11}e_{xx}/K$, where K is the dielectric constant, has a stress $4\pi\epsilon_{11}^2 e_{xx}/K$, which adds to the above-mentioned stress $c_{11}e_{xx}$. Consequently the resultant adiabatic elastic constant c_{11}^0 becomes

$$c_{11}^0 = c_{11} + \frac{4\pi\epsilon_{11}^2}{K} = c_{11}\left\{1 + \frac{4\pi \times (4.77 \times 10^4)^2}{85.45 \times 10^{10} \times 4.5}\right\} = c_{11}(1 + 0.0074). \tag{53}$$

In other words, to replace a quartz oscillating crystal by an equivalent electrical network,[7] we must, as shown in Fig. 4, consider C_1, which represents the capacity of the quartz as a mere dielectric, parallel to the series resonance

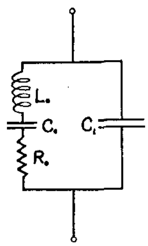

Fig. 4.

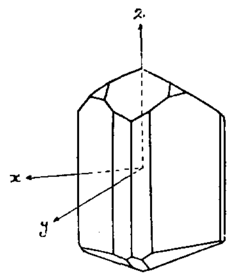

Fig. 5.

circuit L_0, C_0, R_0 representing the mechanical vibrating system. If such is the case, we can easily understand that the electrically measured natural frequency of vibration determined by L_0, C_0, C_1, is always higher than that determined by L_0, C_0. The ratio $C_0 : C_1$, is in general, not greater than 1/100.

[7] D. W. Dye, Piezoelectric quartz resonator and equivalent electrical circuit, Proc. Phys. Soc. London **38**, 399–457 (1926).

IV. Thickness Vibration of Tourmaline Plate

Tourmaline is not yet widely used as a piezoelectric oscillating crystal, and we have no occasion to investigate by fine crystal, but we can foresee the normal modes and frequencies of vibration in the first approximation.

The adiabatic elastic constants referred to the coordinate axes shown in Fig. 5 are composed of the same scheme with (22), the numerical values being

$$\left.\begin{aligned} c_{11} &= 270.17 \times 10^{10}\ \text{dynes/cm}^2, & c_{12} &= 69.06 \times 10^{10}\ \text{dynes/cm}^2,\\ c_{33} &= 160.69 \times 10^{10}\ \text{dynes/cm}^2, & c_{13} &= 8.83 \times 10^{10}\ \text{dynes/cm}^2,\\ c_{44} &= 66.71 \times 10^{10}\ \text{dynes/cm}^2, & -c_{14} &= 7.75 \times 10^{10}\ \text{dynes/cm}^2,\\ && \rho &= 3.100\ \text{grams/cm}^3 \end{aligned}\right\} \quad (54)$$

while the relations between electric fields and strains are:

$$\left.\begin{aligned} e_{xx} &= -d_{22}E_y + d_{31}E_z, & e_{yz} &= d_{15}E_y,\\ e_{yy} &= d_{22}E_y + d_3E_z, & e_{zx} &= d_{15}E_x,\\ e_{zz} &= d_{33}E_z, & e_{xy} &= -2d_{22}E_x,\\ -d_{22} &= 0.69 \times 10^{-8}\ \text{C.G.S. units}, & d_{31} &= 0.74 \times 10^{-8}\ \text{C.G.S. units}\\ d_{33} &= 5.78 \times 10^{-8}\ \text{C.G.S. units}, & d_{15} &= 11.04 \times 10^{-8}\ \text{C.G.S. units} \end{aligned}\right\} \quad (55)$$

(1) If the boundary surfaces of plate be $z=0$ and $z=a$, (20) reduces to

$$\begin{vmatrix} c_{44}-c & 0 & 0 \\ 0 & c_{44}-c & 0 \\ 0 & 0 & c_{33}-c \end{vmatrix} = 0 \quad (56)$$

or

$$\left.\begin{aligned} c_1 = c_2 = c_{44} &= 66.71 \times 10^{10}\ \text{dynes/cm}^2,\\ c_3 = c_{33} &= 160.69 \times 10^{10}\ \text{dynes/cm}^2, \end{aligned}\right\} \quad (57)$$

and the ratio $l:m:n$ corresponding to these values:

$$\left.\begin{aligned} l_1 : m_1 : n_1 &= 1:0:0\\ l_2 : m_2 : n_2 &= 0:1:0\\ l_3 : m_3 : n_3 &= 0:0:1 \end{aligned}\right\} \quad (58)$$

The normal modes of vibration are given by the following equations:

$$\left.\begin{aligned} \xi_1 = u &= A\cos(q\pi x/a)\cdot e^{ipt},\\ \xi_2 = v &= A\cos(q\pi x/a)\cdot e^{ipt},\\ \xi_3 = w &= A\cos(q\pi x/a)\cdot e^{ipt}, \end{aligned}\right\} \quad (59)$$

of which w can only be excited piezoelectrically by the electric field along the axis z, as is seen from (55). Accordingly the vibration is of a pure longitudinal. The corresponding frequency of vibration is

$$p/2\pi = (q/2a)(c_{33}/\rho)^{1/2} = (q/a) \times 0.3600 \times 10^6\ \text{cycles/sec.} \quad (60)$$

(2) If the boundary surfaces of plate be $x=0$ and $x=a$, (8) reduces to

$$\begin{vmatrix} c_{11}-c & 0 & 0 \\ 0 & \frac{1}{2}(c_{11}-c_{12})-c & c_{14} \\ 0 & c_{14} & c_{44}-c \end{vmatrix} = 0 \tag{61}$$

or

$$\left.\begin{aligned} c_1 &= c_{11} = 270.17 \times 10^{10}\ \text{dynes/cm}^2, \\ c_2, c_3 &= \tfrac{1}{4}(c_{11}-c_{12}) + \tfrac{1}{2}c_{44} \pm \left[\left\{\tfrac{1}{4}(c_{11}-c_{12}) - \tfrac{1}{2}c_{44}\right\}^2 + c_{14}^2\right]^{1/2} \\ c_2 &= 102.25 \times 10^{10}\ \text{dynes/cm}^2, \\ c_3 &= 65.03 \times 10^{10}\ \text{dynes/cm}^2, \end{aligned}\right\} \tag{62}$$

and the ratio $l:m:n$ corresponding to these values:

$$\left.\begin{aligned} l_1:m_1:n_1 &= 1:0:0 \\ l_2:m_2:n_2 &= 0:c_2 - c_{44}:c_{14} = 0:-4.59:1 \\ l_3:m_3:n_3 &= 0:c_3 - c_{44}:c_{14} = 0:0.22:1 \end{aligned}\right\} \tag{63}$$

The normal modes of vibration are given by the following equations:

$$\xi = A \cos(q\pi x/a) \cdot e^{ipt} \tag{64}$$

where $\xi_1 = u$, $\xi_2 = -4.59\ v + w$, $\xi_3 = 0.22\ v + w$, while the electric field in the direction x cannot excite the displacement ξ_1 and can hardly excite ξ_2 as is seen from (56), and the periods of ξ_2 and ξ_3 are:

$$p/2\pi = (q/2a)(c_2/\rho)^{1/2} = q/a \times 0.2872 \times 10^6\ \text{cycles/sec.}, \tag{65}$$

$$p/2\pi = (q/2a)(c_3/\rho)^{1/2} = q/a \times 0.2290 \times 10^6\ \text{cycles/sec.} \tag{66}$$