Analysis of Clipping Effect in DMT-based ADSL Systems

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ABSTRACT At present, Discrete Multi-Tone (*DMT*) transceivers receive much attention for the implementation of Asymmetric Digital Subscriber Lines (*ADSL*). This paper analyzes the effect of clipping a *DMT*-signal, i.e. limiting the signal's maximum amplitude. An exact expression is given for the signal-to-noise ratio (*SNR*) degradation due to clipping alone. By combining this expression with the well-known expression for the quantization noise in the *A/D-D/A* converters, it is shown how clipping can reduce the number of bits of *A/D-D/A* converters as well as the dynamical range of the line drivers while keeping the overall *SNR* the same as without clipping.

1. Introduction

Asymmetric Digital Subscriber Lines (*ADSL*) are considered by the majority of the *PTTs* worldwide as a promising technology to increase significantly the (asymmetric) transport capacity of their embedded twisted pair cable infrastructure. In addition to the high rate downstream capacity (and smaller upstream rate), *ADSL* offers the important advantage of being able to frequency multiplex the digital information above a conventional analog voice channel (*POTS*).

Although several modulation line codes can be used to realize such system, the *US ANSI/T1E1.4* standardization Committee has recently selected the *DMT* (Discrete Multi-Tone) line code for *ADSL* access [1]. *DMT* line code divides the total signal power over *N* distinct sub-channels that are each *QAM*-modulated. Since the number of sub-channels (*N*) must be large in order to achieve near optimum performance [2], large amplitude peak values occur in case all sub-channels add constructively. This leads to severe constraints on the dynamical range of the line drivers as well as on the number of bits of the *A/D-D/A* converters. Nevertheless, these spikes occur very rarely, thanks to the statistical averaging over the independent sub-channels.

This paper investigates the effect of limiting the maximum amplitude of the *DMT*-signal (i.e. clipping) on the performance of *ADSL* systems.

Clearly, clipping introduces additional noise because of the loss of information that otherwise would be carried by the clipped portion of the signal. However, as will be seen, this additional noise can be compensated by a reduction of the quantization noise of the *A/D-D/A* converters due to the resulting smaller dynamical range. Moreover, from a certain level of clipping, the number of bits of the *A/D-D/A* converters can further be reduced while keeping the same total *SNR*. Therefore, the same overall performance of the transmission system can be obtained with less severe constraints on the analog part, with easier integration on silicon (*ASIC*) and lower costs as the major benefits.

In Section 2, we briefly review the characteristics of a *DMT* modulated signal. This is followed, in Section 3, by an analysis of the noise introduced by clipping a *DMT*-signal at a predetermined amplitude level. Using this result, we evaluate the influence of clipping on the number of bits of the *A/D-D/A* converters in Section 4. In all these sections the following assumptions are made:

1. The power spectrum of the clipped portion of the signal is flat and fully contained within the same bandwidth as the useful signal.
2. The total signal power is uniformly distributed over all carriers.
3. All carriers have the same *QAM* constellation size.
4. All power calculations use a normalized impedance of one Ohm.

The effect of different *QAM* constellation sizes is analysed in Section 5, while the spreading of part of the clipping noise power spectrum outside the *DMT*-signal bandwidth is analysed in Section 6. Finally, conclusions are given in Section 7. Two appendices are added at the end.

2. Background

A *DMT* modulated signal is basically the sum of *N* independently *QAM* modulated signals (sub-channels), each carried over a distinct carrier [3]. The frequency separation of the *N* carriers is $1/T$ where *T* is the time duration of a *DMT*-symbol . An example of a *DMT*-symbol with $N=256$ is shown in Fig. 1. It is seen that the instantaneous amplitude, $A(t)$, spans a large range with a maximum value that arises when the *N* modulated signals add in-phase. When *N* is large, $A(t)$ can

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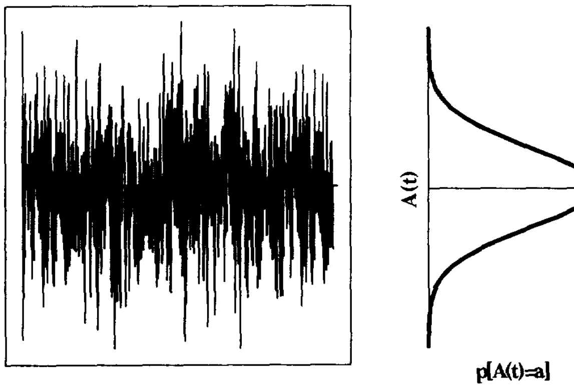
be accurately modelled by a Gaussian random process (Central limit theorem [4]) such that the probability densityfunction of *A(t)* is Gaussian-shaped as shown in Fig.1.b.



(a) *(b)*

Figure 1 *Instantaneous amplitude A(t) of a DMT-symbol (N=256) and its associated probability distribution*

Since the maximum instantaneous amplitude, A_{max}, is expected to arise very rarely (statistical averaging) it might be advantageous to accept some clipping in the *DMT*-symbol and to trade-off the resulting *SNR* loss against the gain of signal-to-quantization noise ratio of the *A/D-D/A* converters and line driver dynamical ranges obtained with a reduced signal amplitude. It will be shown that even without loss of the *SNR*, clipping can reduce the number of bits of the *A/D-D/A* converters. In other words, while clipping introduces additional noise, the decrease of the dynamical range allows a reduction of the number of bits for the *A/D-D/A* converters while keeping the same overall *SNR* ratio as when clipping is not applied.

3. Analysis of clipping noise

The computation of the frequency spectrum of the clipped portion of the signal is arduous. However, an analytical expression of the clipping-noise can be obtained.

A similar problem of clipping has been analysed by *A. Saleh* [5] in the framework of fiber optic subcarrier multiplexing systems. An approximate analysis of clipping in the context of *DMT*-based *ADSL* systems has been done by the authors [6]. Further investigation has shown that the analysis can be done exactly and is the subject of the present paper.

As noticed in Section 2, for large values of N ($N \geq 10$), $A(t)$ can be modelled by a Gaussian random process with a zero mean value and a variance σ^2. That is, the probability that, at any given time, $A(t)$ takes the value x is given by:

$$\Pr ob\{A(t)=x\} = p(x) = \frac{1}{\sigma\sqrt{2\pi}}.\exp\left[-\frac{x^2}{2\sigma^2}\right] \quad (1)$$

If the maximum absolute amplitude of the *DMT*-symbols is limited to A_{clip}, the total power of the clipped portion is given by (see appendix 1 for the details of calculation):

$$P_{Clip} = 2.\int_{A_{Clip}}^{+\infty}(x - A_{Clip})^2.p(x)dx$$

$$= \sigma^2.\left(-\sqrt{\frac{2}{\pi}}\mu.e^{-\frac{\mu^2}{2}} + (1+\mu^2).erfc\left(\frac{\mu}{\sqrt{2}}\right)\right) \quad (2)$$

where $\mu = A_{Clip}/\sigma$.

Since the total signal power is $P_{Tot} = \sigma^2$, the signal-to-clipping noise power ratio is readily obtained using Eq.(2):

$$(S/N)_{Clip} = \left((1+\mu^2).erfc\left(\frac{\mu}{\sqrt{2}}\right) - \sqrt{\frac{2}{\pi}}.\mu.e^{-\frac{\mu^2}{2}}\right)^{-1} \quad (3)$$

Fig. 2 shows SNR_{Clip} (in dB) as a function of μ. For example, if we impose $SNR_{Clip} \geq 50dB$, Eq.(3) provides $\mu \geq 3.89$.

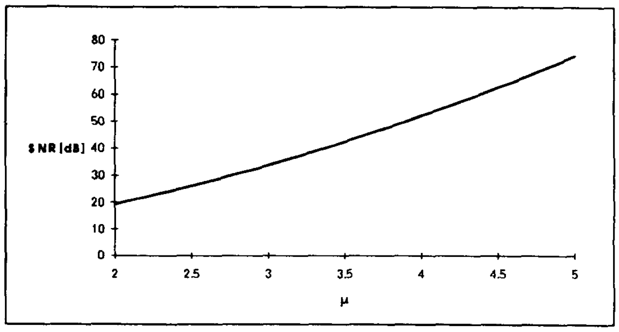


Figure 2 *SNR_{Clip} [dB] as a function of $\mu = A_{Clip}/\sigma$.*

For a given SNR_{clip} and a given transmit power, the value of A_{Clip} can be determined using Eq.(3). The gain in dynamical range obtained by clipping is given by A_{max}/A_{Clip}. To evaluate this gain, we derive an expression for A_{max} which depends upon the number of sub-channels *N*. For simplicity we first assume identical *QAM* constellation sizes for all carriers. The case of different *QAM* constellation sizes will be investigated in Section 5. The average power for each sub-channel is given by [7]:

$$\overline{P_n} = \frac{a^2}{3}.(L^2 - 1) \quad (4)$$

where *2a* is the distance between two neighbouring points and L^2 is the number of points within the *QAM* constellation.

Assuming a uniform power distribution amongst all sub-channels (cf. water pouring algorithm: $\overline{P}_n = \sigma^2 / N$), the value of a is obtained from Eq.(4) as:

$$a^2 = \frac{3\sigma^2}{N(L^2-1)} \qquad (5)$$

While the maximum amplitude of each sub-channel equals $\sqrt{2}.a.(L-1)$, the maximum instantaneous amplitude of the *DMT*-symbol can be as high as $[(1+\sqrt{2}).N.a.(L-1)/2]$ for large values of N (see appendix 2). Therefore with Eq.(5) we obtain:

$$A_{max} = \frac{1+\sqrt{2}}{2}\sigma\sqrt{3N}.\sqrt{\frac{L-1}{L+1}} \qquad (6)$$

and the ratio A_{max}/A_{Clip} is thus given by:

$$r(\mu,N,L) = \frac{A_{max}}{A_{Clip}} = \frac{1+\sqrt{2}}{2}.\frac{\sqrt{3N}}{\mu}.\sqrt{\frac{L-1}{L+1}}$$

$$\cong 2\frac{\sqrt{N}}{\mu}.\sqrt{\frac{L-1}{L+1}} \qquad (7)$$

Fig. 3 plots the ratio A_{max}/A_{Clip} expressed in dB for L=4 (16-*QAM*) and $\mu=\mu_0$ =3.89 (SNR_{Clip}=50dB).

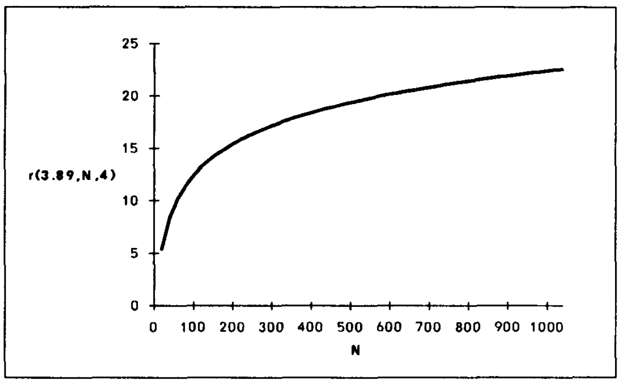


Figure 3 *20 log$_{10}$r as a function of N for $\mu=\mu_0$=3.89 and L=4 (16-QAM sub-channels)*

It is seen that r=16.46 dB (ratio 6.65) for N=256, and r=19.47 dB (ratio 9.4) for N=512. As seen from Eq.(7), r increases proportionally with $\sqrt{N}$. Note that the total number of transmitted bits per symbol increases linearly with N in case L is fixed. If the total number of bits (B) is fixed, it is more relevant to express the ratio A_{max}/A_{Clip} in function of B by taking into account the relation between the constellation size and the number of sub-channels. In this case, Eq.(7) becomes:

$$r(\mu,N,B) = \frac{A_{max}}{A_{Clip}} = \frac{1+\sqrt{2}}{2}.\frac{\sqrt{3N}}{\mu}.\sqrt{\frac{2^{\frac{B}{2.N}}-1}{2^{\frac{B}{2.N}}+1}} \qquad (8)$$

Fig. 4 plots the ratio A_{max}/A_{Clip} expressed in dB for B=1024 and $\mu=\mu_0$=3.89 (SNR_{Clip}=50dB).

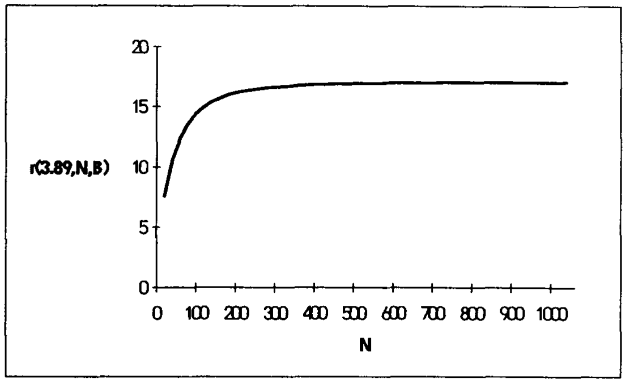


Figure 4 *20 log$_{10}$r as a function of N for μ_0=3.89 and B=1024*

Note that, in contrast to Fig.3, as N increases, r approaches asymptotically the value r_{limit}=17.086 dB (ratio 7.15). Moreover, for N=256, the value of r is very close to this asymptotic value.

4. Influence on A/D-D/A converters

The previous sections analyse the *SNR* degradation due to clipping alone. To estimate its effect on the required number of bits of the *A/D-D/A* converters one must combine Eq.(3) with the *SNR* due to quantization.

If there is no clipping, the number of bits of the *A/D-D/A* converters, R_1, is determined by the dynamical range of the signal and the required signal-to-quantization noise ratio, $(S/N)_{Q1}$.

Suppose now that the signal is clipped when $|A(t)|>A_{Clip}$ such that the maximum value of $|A(t)|$ is equal to A_{Clip}. This reduces the dynamical range of the signal. Consequently, the required number of bits , R_2, of the *A/D-D/A* converters needed to keep the same signal-to-noise ratio as $(S/N)_{Q1}$ is expected to be smaller than R_1. However, since clipping introduces its own noise, R_2 will be determined according to:

$$(S/N)^{-1}_{Q1} = (S/N)^{-1}_{Q2} + (S/N)^{-1}_{Clip} \qquad (9)$$

With this requirement, the overall *SNR* due to both clipping and quantization will keep the same value as $(S/N)_{Q1}$. The signal-to-quantization noise ratio can be expressed as [7]:

$(S/N)_Q = \frac{12\sigma^2 2^{2R}}{(2\hat{A})^2}$ where $\hat{A}$ is the maximum amplitude of the signal and R is the number of bits of the *A/D-D/A* converter. Introducing this expression into Eq.(9), we obtain after a few arithmetic's:

$$\nu^2 - 2^{2\Delta}\mu^2 = 3.2^{2R_1}.\left((1+\mu^2).erfc\left(\frac{\mu}{\sqrt{2}}\right) - \sqrt{\frac{2}{\pi}}.\mu.e^{-\frac{\mu^2}{2}}\right) \quad (10)$$

where $\mu = A_{Clip}/\sigma$, $\nu = A_{max}/\sigma$ and $\Delta = R_1 - R_2$.
The parameter ν is fixed by N and L following Eq.(6): $\nu = \frac{1+\sqrt{2}}{2}.\sqrt{3N}.\sqrt{\frac{L-1}{L+1}}$. Solving Eq.(10) for Δ, we obtain:

$$\Delta = \frac{1}{2}.log_2\left\{\frac{\nu^2 - 3.2^{2R_1}.\left((1+\mu^2).erfc\left(\frac{\mu}{\sqrt{2}}\right) - \sqrt{\frac{2}{\pi}}.\mu.e^{-\frac{\mu^2}{2}}\right)}{\mu^2}\right\} \quad (11)$$

Eq.(11) expresses the gain (obtained by clipping) in the required number of bits of the *A/D-D/A* converters as a function of the two parameters R_1 and μ.
Fig. 5 shows Δ as a function of μ for different values of R_1 and for $N=256$ and $B=1024$. As can be seen from Eq. (11), the argument of $log_2 \{...\}$ is zero for a specific value of μ. This point corresponds with the value of A_{Clip} for which the clipping noise equals the quantization noise N_{Q1}. At this point R_2 should be infinite.
As μ increases, $(S/N)_{Clip}$ increases rapidly (cf. Fig. 2) which results in a fast decrease of $(S/N)_{Q2}$ and thus of the required number of bits R_2. For a certain value of μ which depends upon R_1 (μ_{min}), a break-even point is reached at which $\Delta=0$; $\mu_{min}=3.17$, 3.83, 4.42, and 4.95 for $R_1=10$, 12, 14 and 16 bits, respectively.
For greater values of μ (and thus of A_{Clip}), a maximum is reached for $\mu=\mu_{opt}$. In order to have $\Delta \cong \Delta_{max}$, μ should only be increased by a few percent compared to μ_{min}, owing to the steepness of $\Delta(\mu)$ in the vicinity of μ_{min}. At this point a significant reduction of the number of bits is obtained which results in a reduction of the implementation complexity of the *A/D-D/A* converters especially at high clock rates.
As μ further increases from μ_{opt}, the resulting increase of the amplitude range ($2.A_{Clip}$) dominates over the decrease of $(S/N)_{Q2}$, which results in the negative slope of the Δ-curve as seen on Fig. 5. A second break-even point is reached at $A_{Clip}=A_{max}$ for which $(S/N)_{Clip}=\infty$. This is at $\mu=25.9$ for $N=256$. Clearly, this point is independent of R_1. Therefore all curves in Fig. 5 converge for $\mu \geq 6$.

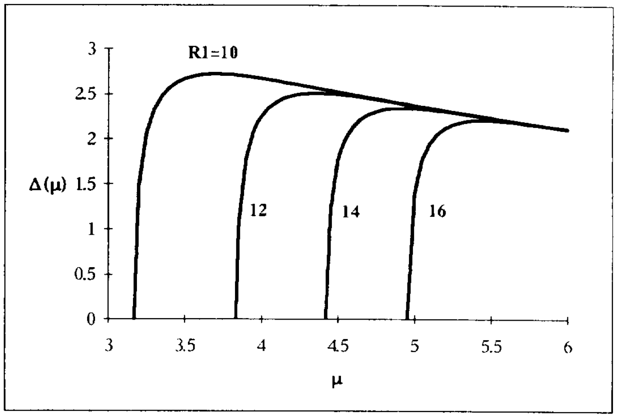


Figure 5 $\Delta = R_1 - R_2$ *as a function of* μ *for* $N=256$, $B=1024$ *and* $R_1=10$, *12, 14 and 16 bits*

It is also seen from Fig. 5 that as the number of bits R_1 increases, the value of μ for which, e.g., $\Delta=2$ increases while the maximum gain, Δ_{max}, as well as the range of μ for which $\Delta \geq 2$ decrease.
Fig. 6 depicts the relation between μ_{opt} and R_1 for $N=256$ and 512. It is seen that μ_{opt} increases quasi-linearly with R_1 and is almost independent of N. Note that for $\mu_{opt} \leq \mu \leq 6.5$, $\Delta \geq 2$ (cf. Fig. 5).

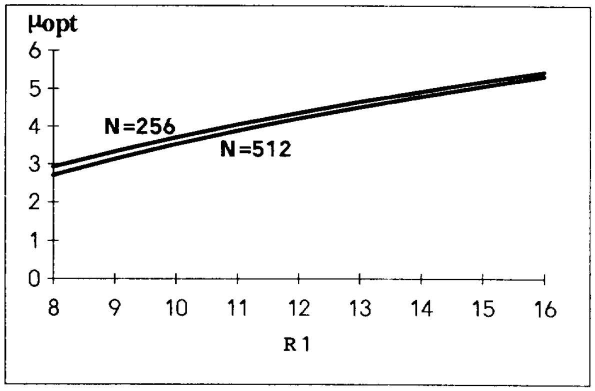


Figure 6 *Optimum values of the clipping parameter* μ_{opt} *in function of* R_1 *for* $L=4$ *(16-QAM),* $N=256$ *and 512*

Finally, Fig. 7 shows the dependency of Δ_{max} as a function of N for $R_1=12$ and 16 bits. The implementation complexity of the *DMT* transceiver (*FFT*) limits the number of sub-channels (N). For realistic values of N ($256 \leq N \leq 1024$), Δ_{max} is bounded by $2 \leq \Delta_{max} \leq 3$.

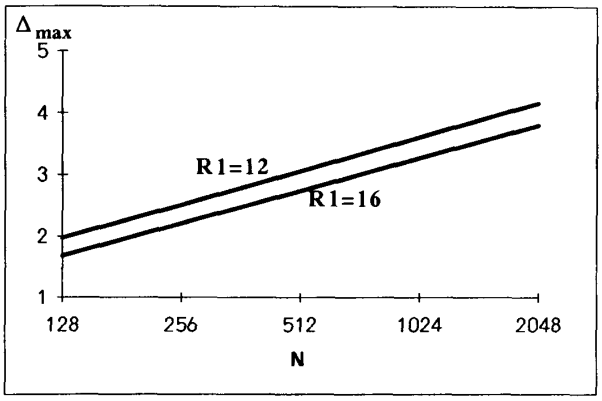


Figure 7 *Δ_{max} as a function of N for R_1=12 or 16 bits and L=4 (16-QAM)*

5. Variation of constellation sizes

The above analysis assumes identical *QAM*-constellation sizes for all sub-channels in the *DMT*-signal. In the present section, we investigate the effects of a variation of the constellation sizes.

First we study the effect of assigning zero bit (and zero power) to *K* out of *N* sub-channels. This is the case when the *SNR* at those sub-channels is below a given threshold (because of crosstalk from other services or an impairment on the line - such as the presence of a bridged tap - that introduces severe disturbance at these frequencies). If the *B* bits per symbol are still uniformly distributed over the remaining *(N-K)* carriers, the expression for A_{max}, Eq.(6), remains valid provided that *N* is replaced by *(N-K)*. This also holds for the related expression for A_{max}/A_{Clip}, Eq.(7), with $L = 2^{\frac{B}{2.(N-K)}}$. Therefore Eq.(8) becomes:

$$r(\mu, N, B) = \frac{A_{max}}{A_{Clip}} = \frac{1+\sqrt{2}}{2} \cdot \frac{\sqrt{3(N-K)}}{\mu} \cdot \sqrt{\frac{2^{\frac{B}{2.(N-K)}}-1}{2^{\frac{B}{2.(N-K)}}+1}} \qquad (12)$$

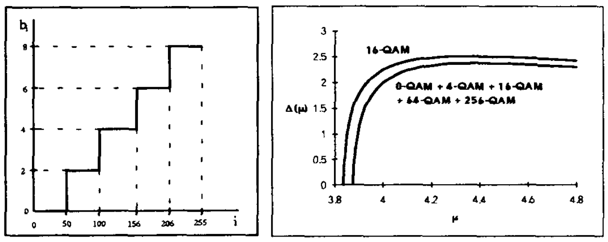
The gain in the number of bits for the *A/D-D/A* converters is still given by Eq.(11) but with the parameter ν modified according to Eq.(12), i.e. $\nu = r.\mu$. It turns out that if half of the sub-channels are put to zero while keeping the same value of *B*, the relevant values of μ and Δ and thus of the allowed clipping level, only undergo minor changes. For example, with R_1=*12* bits, *N=256*, *K=128* and *B=1024*, μ_{min} moves from *3.836* to *3.906*, μ_{opt} from *4.354* to *4.422* and Δ_{max} from *2.509* to *2.267*.

In the previous example we allocated the same number of bits to the used sub-channels. However, due to the frequency dependency of twisted pair capacity [8] as well as the possible interferers, it is advantageous to use larger constellation sizes at frequencies where the *SNR* is high. To evaluate the effect of different constellation sizes, the *N* sub-channels are divided into *M* sub-groups of c_j sub-channels each (*j=0..M-1*), the sub-groups having a fixed but different number of bits per carrier: $b_j=2.j$ (only square constellations are considered). Taking into account the fact that the total number of bits *B* is fixed and still supposing a uniform power distribution across the sub-channels [9], Eq.(6) now becomes:

$$A_{max} = \frac{1+\sqrt{2}}{2} \sigma \sqrt{\frac{3}{N-s_0}} \cdot \sum_{j=1}^{M-1} c_j \cdot \sqrt{\frac{2^j-1}{2^j+1}} \qquad (13)$$

where $\sum_{j=0}^{M-1} c_j = N$ and $\sum_{j=0}^{M-1} c_j . b_j = B$.

Δ given by Eq.(11) still applies provided that ν is adapted according to Eq.(13). Fig. 8b shows the difference between a fixed constellation size (*16-QAM*) and the quasi-uniform repartition given in Fig. 8a. Again, there are only minor changes for the relevant values of μ and Δ. For example, μ_{min} moves from *3.836* to *3.875*, μ_{opt} from *4.354* to *4.392* and Δ_{max} from *2.509* to *2.374*.



(a) (b)

Figure 8 *Δ=R_1-R_2 as a function of μ for R_1=12 bits, B=1024, N=256 and constellation sizes fixed (16-QAM) or given by Fig. 8a*

We conclude that the results obtained with constant constellation sizes are good approximations to those found with non-uniform bit assignments.

6. Spreading of the clipping noise outside the *DMT*-signal bandwidth

In all preceding sections, we assumed that the clipping noise power spectral density is flat and uniformly distributed inside the *DMT*-signal bandwidth. However, part of the clipping power spectrum falls outside the bandwidth of the *DMT*-signal (i.e. the frequency band occupied by the modulated carriers), and can thus be filtered out.

In [10], Mazo calculates the spectrum of the signal distortion in case of *CATV* transmission over optical fibers. If Γ is the fraction of the clipping noise power that falls inside the signal bandwidth (Γ<1), filtering improves the SNR_{Clip} of Eq.(3) by a factor Γ^{-1}. Mazo shows that over a wide range of μ, the correction factor Γ is almost constant and is largely dependent on the carrier frequency allocation of the useful signal. The analysis of Mazo can be extended to the context of *DMT*-systems with a frequency range from 10 kHz to 1 MHz. This leads to $\Gamma \approx 0.68$.

Taking the factor Γ into account, Eq.(11) becomes:

$$\Delta = \frac{1}{2}.log_2 \left\{ \frac{\nu^2 - 3.2^{2R_1}.\Gamma.\left((1+\mu^2).erfc\left(\frac{\mu}{\sqrt{2}}\right) - \sqrt{\frac{2}{\pi}}.\mu.e^{-\frac{\mu^2}{2}} \right)}{\mu^2} \right\} \quad (14)$$

Eq.(14) shows the gain in the number of bits, Δ, when part of the clipping noise power is spread outside the *DMT*-signal bandwidth. Obviously, the case $\Gamma=1$ represents the worst case and all the preceding results will improve for $\Gamma<1$.

Fig. 9 shows the difference for the gain Δ with $\Gamma=1$ and $\Gamma=0.68$ for $R_1=12$ bits, $N=256$, and $B=1024$. It is seen that there are only slight modifications of the relevant values of μ and Δ. For example, μ_{min} moves from *3.836* to *3.745*, μ_{opt} from *4.354* to *4.266* and Δ_{max} from *2.509* to *2.536*. The fact that μ decreases and Δ_{max} increases for $\Gamma<1$ is related to the improvement of SNR_{clip}.

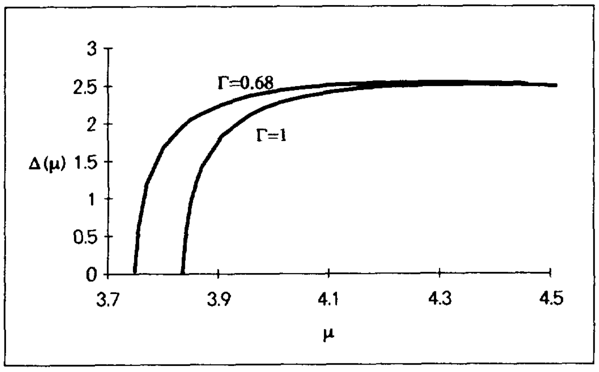


Figure 9 $\Delta=R_1-R_2$ *as a function of* μ *for* $R_1=12$ *bits,* $B=1024$*,* $N=256$*,* $\Gamma=1$ *and 0.68*

6. Conclusions

An exact expression of the distortion introduced by clipping a *DMT*-signal has been derived. This expression has been used to determine analytically the gain, Δ, in the number of bits of the *A/D-D/A* converters while keeping the same performance as if clipping was not applied. It also allows to accurately determine the clipping level for a given *A/D-D/A* precision. We investigated the effect of various sub-channel bit allocation schemes and the spectral spread of the clipping noise power. It is found that the simple case of uniform constellation sizes and all noise power in-band provides a good estimate for Δ and for the permitted clipping level.

The main conclusion is that clipping allows to reduce the implementation complexity of *DMT*-based *ADSL* systems. For example, with realistic assumptions for the number of sub-channels ($N=256$), the number of bits of the *A/D-D/A* converters can be decreased by more than *2* bits with a clipping amplitude $A_{Clip} \cong A_{max}/6$ while keeping the same performance as if clipping was not applied.

Appendix 1

Eq.(2) is obtained by remarking that P_{Clip} can be expressed as:

$$P_{Clip} = \frac{2}{\sigma.\sqrt{2.\pi}} \int_{A_{Clip}}^{+\infty} (\sigma^4 d_x^2 + 2A_{Clip}\sigma^2 d_x + \sigma^2 + A_{Clip}^2) e^{-\frac{x^2}{2\sigma^2}} dx \quad \text{(A.1.1)}$$

where d_x and d_x^2 stand for the single and second derivatives with respect to x. The integration of the first two terms of the integrand is straightforward and gives the first term in brackets of Eq.(2). For the last two terms of Eq.(A.1), the change of variable $x = \sqrt{2}.\sigma.y$ leads naturally to the error function *erfc(t)* defined by:

$$erfc(t) = \frac{2}{\sqrt{\pi}}.\int_t^{\infty} e^{-y^2}.dy \quad \text{(A.1.2)}$$

and the second term in brackets of Eq.(2) is therefore immediately obtained.

Note that an approximate expression for P_{Clip} can be obtained by a development of the *erfc* function [11]:

$$P_{Clip} \approx 2.\sqrt{\frac{2}{\pi}}.\sigma^2.\mu^{-3}.e^{-\frac{\mu^2}{2}}.\left(1+\frac{6}{\mu^2}\right)^{-1} \quad \text{(A.1.3)}$$

This approximation is accurate to within *3%* for $\mu \geq 3$.

Appendix 2

In this appendix, the maximum amplitude of a *DMT*-signal is determined.

For one symbol duration, the amplitude of a *DMT*-signal is given by:

$$A(t) = \text{Re}\left\{ \sum_{k=0}^{N-1} c_k.e^{-j2\pi k\frac{t}{T}} \right\} \qquad 0 \leq t \leq T \quad \text{(A.2.1)}$$

where T is the duration of the *DMT*-symbol, c_k is the chosen point in the constellation of the k-th carrier and N is the number of carriers in the *DMT*-system. The *DMT*-signal can thus be considered as the sum of N independent *QAM* signals with frequencies that are multiples of $1/T$. In case of baseband transmission, the carrier $k=0$ (dc) is not used.

It is assumed that all carriers have a square constellation as depicted in Fig. A.2.1 and that they all have the same maximum amplitude $\sqrt{2}$ (the square is "normalized" to simplify the calculations).

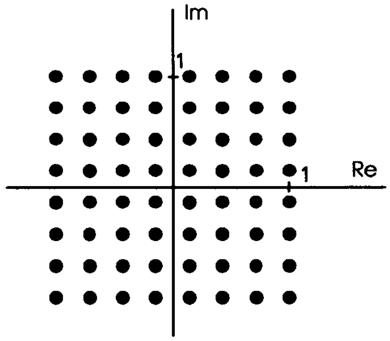


Figure A.2.1 *Points of a 64-QAM square constellation*

Remark that the maximum amplitude of the *DMT*-signal is uplimited by $N\sqrt{2}$. Note that this upper limit is reached (at $t=0$) in case all constellations are rotated over $\pi/4$. However, this choice is unusual as it results in a larger maximum amplitude than the one coming from the constellation in Fig. A.2.1 ($N.\sqrt{2}$ instead of $N.\frac{\sqrt{2}+1}{2}$ as will be apparent later) for the same average power.

Clearly, each choice of N points (c_k) leads to a certain maximum amplitude over the duration of one symbol. Therefore the problem is to find out which combination of points gives rise to the overall maximum amplitude A_{max}.

The main idea to find this combination is to use the image of rotating vectors. Each carrier k can be considered as a rotating vector whose angular velocity equals k/T. The chosen point in the constellation (c_k) defines the vector at time $t=0$. The contribution of each carrier is given by the projection of the corresponding vector on the real axis. $A(t)$ is the sum of all these projections.

Due to the symmetry of the square constellations, one only needs to consider the positive real axis. Moreover, it is sufficient to consider only the interval $0 \le t \le \frac{T}{8}$. Indeed, for $\frac{T}{8} \le t \le \frac{T}{4}$ and the subsequent intervals, the following reasoning can be repeated with other choices for the selected points (c_k), but resulting in the same maximum amplitude.

It is obvious that for any constellation, at any time t, the point that realizes the maximum contribution (projection on the positive real axis) is a corner point. Therefore, it is only necessary to study the corner points.

The choice for odd values of k is evident: $c_k = \sqrt{2}.e^{jk.\frac{\pi}{4}}$. For k even, the corner that gives the greatest contribution for $0 \le t \le \frac{T}{8}$ is $c_k = \sqrt{2}.e^{j.(k-1).\frac{\pi}{4}}$ (see figure A.2.2).

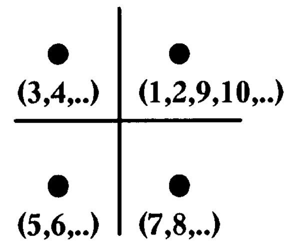


Figure A.2.2 *Choice for the maximum amplitude points*

With this choice we easily obtain $A\left(\frac{T}{8}\right) = \frac{1+\sqrt{2}}{2}N$ and therefore $\frac{1+\sqrt{2}}{2}N \le A_{max} \le \sqrt{2}N$.

It is now possible to compute A_{max} with the previous selected points. For simplicity, we will consider the case where N is even. Taking into account the above expressions for the coefficients c_k, straightforward algebra leads to the following equation for the amplitude:

$$A(t) = 2.\sqrt{2}.\cos\left(\pi.\frac{t}{T}\right).\cos\left\{\pi.\left(\frac{t}{T}-\frac{1}{8}\right).(N+1)+\frac{\pi}{8}\right\}.\frac{\sin\left\{\pi.\left(\frac{t}{T}-\frac{1}{8}\right).N\right\}}{\sin\left\{2.\pi.\left(\frac{t}{T}-\frac{1}{8}\right)\right\}} \qquad (A.2.2)$$

It is easy to check that for $t=T/8$, we obtain the same result for $A(t)$ as Eq.(6).

To determine the maximum amplitude, it is necessary to compute the time t_{max} that maximizes Eq.(A.2.2). This leads to an equation for t obtained by derivating Eq.(A.2.2) with respect to t. Unfortunately, it is not possible to solve this equation analytically and only an approximation for t_{max} can be obtained.

For this purpose, we make the following assumption on t_{max}: $t_{max}=(1/8+x)T$ where x behaves like y/N and y is small. This means that we anticipate that for large values of N, t_{max} approximates $T/8$. Consistency of this assumption is obtained if the expression chosen for t_{max} effectively leads to a small value for y when included in the equation that determines t_{max}. This is indeed the case and one gets in first order approximation:

$$y = \frac{-3}{4.\left(1+\sqrt{2}\right)} \cdot \frac{1}{1+\frac{1-2\sqrt{2}}{1+\sqrt{2}}.\frac{1}{N}+\frac{3\sqrt{2}}{1+\sqrt{2}}.\frac{1}{N.(N+2)}} \quad \text{(A.2.3)}$$

Substitution of this value in the expression for t_{max} leads to:

$$t_{max} \approx \frac{T}{8} - \frac{3.T}{4.\pi.\left(1+\sqrt{2}\right).N} \quad \text{(A.2.4)}$$

Taking the first order approximation of Eq.(A.2.2) and evaluating this expression for $t=t_{max}$ gives:

$$A_{max} \approx \frac{1+\sqrt{2}}{2}.N.\left[1+\frac{3}{4.\left(1+\sqrt{2}\right)^2}+O\left(\frac{1}{N}\right)\right] \quad \text{(A.2.5)}$$

The deviation of this value from the maximum amplitude used in Eq.(6) is *13%* . More precise results, obtained by a second order approximation, lead to a deviation of *6%* :

$$A_{max} = \frac{1+\sqrt{2}}{2}.N.1,064 \quad \text{(A.2.6)}$$

This was confirmed by numerical simulations as shown in figure A.2.3 which depicts the ratio R between A_{max} and $N.\frac{\sqrt{2}+1}{2}$.

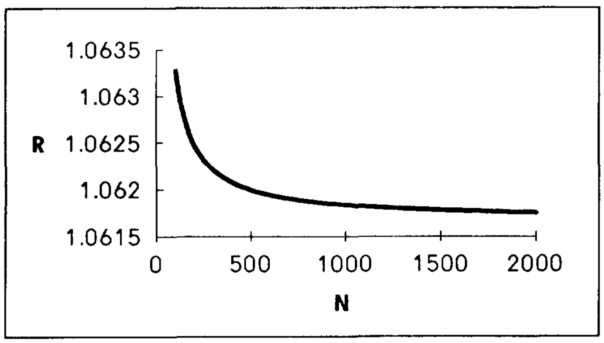


Figure A.2.3 *Ratio between* A_{max} *and* $N.\frac{\sqrt{2}+1}{2}$ *in terms of N*

Through simulation, it was verified that the *6%* deviation of A_{max} from $N.\frac{\sqrt{2}+1}{2}$ only introduces small deviations for the results obtained in the main text.

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